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C4 chapter 3 – The Binomial Theorem

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Note Title 15/02/2011

We know that we can expand $(1+x)^3$ in at least 3 ways:

① multiply out $(1+x)(1+x)(1+x)$

$$= (1+x)(1+2x+x^2)$$

$$= 1 + 2x + x^2$$
$$+ x + 2x^2 + x^3$$

$$= 1 + 3x + 3x^2 + x^3$$

② use Pascal's triangle

1

1 1

1 2 1

1 [illegible] 3 1

to predict coefficients: $\underline{1} + \underline{3}x + \underline{3}x^2 + \underline{1}x^3$

③ Use combinations to predict coefficients

$$(1+x)(1+x)(1+x)$$

- how many ways can I pick 3 ones & no x's?
  only 1 way: so $1(1)^3(x)^0 = 1$
  writing ${}^3C_1 (1)^3 (x)^0 = 1$

- how many ways can I pick 2 ones & 1 x
  3 ways: it could be from any of 3 brackets
  write ${}^3C_1 (1)^2 (x)^1 = 3x$

- how many ways can I pick 1 one & 2 x's

3 ways to pick the first x
2 ways left to pick the second x
leaving only 1 way to pick the 1
However there aren't 6 ways $(3 \times 2 \times 1)$

altogether because

$$(1+x_1)(1+x_2)(1_3+x)$$

and

$$(1+x_2)(1+x_1)(1_3+x)$$

... I've double counted these because there's more than one way to pick x from brackets A & B but not C

So $$^3C_2 = \frac{3 \times 2 \times 1}{\boxed{2 \times 1} \times \boxed{1}}$$

the choices above

ways to arrange x's

ways to arrange the 1's

This explains $$^nC_r = \frac{n!}{(n-r)!\,r!}$$

where I choose r items from n

- how many ways to choose 3 x's — 1

Check that $$^3C_3 = \frac{3!}{(3-3)!\,3!} = \frac{6}{0! \times 6}$$

so the logical conclusion is that $0! \equiv 1$

- how many ways to choose 4 x's from the above?

$$^3C_4 = \frac{3 \times 2 \times 1 \times 0}{(1) \times (4 \times 3 \times 2 \times 1)} = 0$$

$\Rightarrow$ in the expansion of $(1+x)^3$ there are $0x^4$

... and $0x^5 + 0x^6 + \ldots$

$\Rightarrow$ This is why $(1+x)^3 = 1 + 3x + 3x^2 + x^3$

because we know there are no more.

(M) What if we wanted $(1+x)^{-2}$ or $(1+x)^{1/2}$

$$(1+x)^{-2} = 1 + \frac{(-2)(x)^1}{1} + \frac{(-2)(-3)(x)^2}{2!} + \frac{(-2)(-3)(-4)(x)^3}{3!} + \frac{(-2)(-3)(-4)(-5)(x)^4}{4!}$$

$+ \ldots$ until you get the zero which tells you to stop.

but you never get to zero so it never stops!

so the expansion is infinitely long with higher & higher powers of x.

how about

$$(1+x)^{1/2} = 1 + \frac{\left(\frac{1}{2}\right)(x)^1}{1!} + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(x)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(x)^3}{3!} + \ldots$$

but you never get to zero so it never stops!

so the expansion is infinitely long with higher & higher powers of x.

(M2) So is it all broken? A load of nonsense?

certainly if $x > 1$ x^3, x^4, x^5 get bigger & bigger so the stuff we've missed off is really important.

indeed if $x < -1$ x^3, x^4, x^5 get further away from zero so it will flip-flop positive, negative etc
Also useless

but when $-1 < x < 1$ ie x is small then
$|x| > |x^2| > |x^3| > |x^4|$ ...

So the powers get closer & closer to zero... & sooner or later they get so small they really don't matter.

... and the smaller x is the faster it converges.

(M3) examples find first four terms of ...

A
$$\frac{1}{1+x} = (1+x)^{-1} = 1 + \frac{(-1)(x)}{1!} + \frac{(-1)(-2)(x)^2}{2!} + \frac{(-1)(-2)(-3)(x)^3}{3!} + \dots$$
$$= 1 - x + x^2 - x^3 + \dots$$

provided x is small ie $|x| < 1$

B
$$\sqrt{1-3x} = (1-3x)^{1/2} = 1 + \frac{(\frac{1}{2})(-3x)^1}{1!} + \frac{(\frac{1}{2})(\frac{-1}{2})(-3x)^2}{2!} + \frac{(\frac{1}{2})(-\frac{1}{2})(\frac{-3}{2})(-3x)^3}{3!} + \dots$$
$$= 1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 + \dots$$

.. provided $-3x$ is small ie $|-3x| < 1$
$\Rightarrow |x| < \frac{1}{3}$

C
$$(1-x)^{1/3} = 1 + \frac{(\frac{1}{3})(-x)^1}{1!} + \frac{(\frac{1}{3})(\frac{-2}{3})(-x)^2}{2!} + \frac{(\frac{1}{3})(\frac{-2}{3})(\frac{-5}{3})(-x)^3}{3!} + \dots$$

$$= 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 + \dots \qquad |x| < 1$$